WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2016
TEST 3: Derivatives and Integrals

Name: ________________________

Monday 15th August Time: 45 minutes Mark /40

Section 1 – Calculator free 20 marks

1. [4 marks]

A curve is defined by the equation $ye^x + xy^2 = 1$

Use implicit differentiation to determine the equation of the normal drawn at the point $(0,1)$.

$$\frac{dy}{dx}\cdot e^x + y \cdot e^x + y^2 + 2xy\frac{dy}{dx} = 0 \quad ✓✓$$

$$(0,1) \Rightarrow \frac{dy}{dx} + 1 + 1 = 0$$

$$\frac{dy}{dx} = -2 \quad ✓$$

$\Rightarrow$ normal is $y = \frac{x}{2} + 1$ ✓

2. [3 marks]

Determine the area enclosed by the curves $y = x+1$ and $y = x^2+1$, as shown.

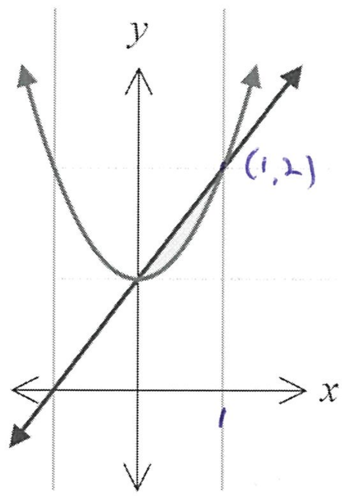
$$A = \int_0^1 x+1-(x^2+1)\,dx \quad ✓$$

$$= \int_0^1 x - x^2\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 \quad ✓$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ units}^2 \quad ✓$$



3. [13 marks – 4, ~~4~~, ~~1~~ and 4] 1 4

Determine each of the integrals given

(a) $\int_0^2 3\pi x^2 (x^3 - 1)\, dx$ by using a suitable substitution

$t = x^3 - 1$

$dt = 3x^2\, dx$ ✓

$x = 0,\ t = -1$

$x = 2,\ t = 7$ ✓

$\therefore \int = \int_{-1}^{7} \pi . t \; dt$ ✓

$= \left. \frac{\pi t^2}{2} \right|_{-1}^{7}$

$= \frac{49\pi}{2} - \frac{\pi}{2} = 24\pi$ ✓

(b) $\int \sin 3x \cos^2 3x\, dx$

$= \frac{-\cos^3 3x}{9} + c$ ✓

(c) $\int \sin^4 x\, dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$ ✓

$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \; dx$ ✓

$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right) dx$ ✓

$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$ ✓

(d) $\int \frac{dx}{x^2 - 1}$ in the form $\ln A + c$

$\frac{1}{x^2 - 1} = \frac{a}{x-1} + \frac{b}{x+1}$ ✓

$= \frac{ax + a + bx - b}{(x-1)(x+1)}$

$\Rightarrow a = -b$

$a - b = 1$

$\Rightarrow a = \frac{1}{2} \quad b = -\frac{1}{2}$ ✓

$= \int \frac{\frac{1}{2}}{x \mp 1} + \frac{-\frac{1}{2}}{x+1} \; dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$ ✓

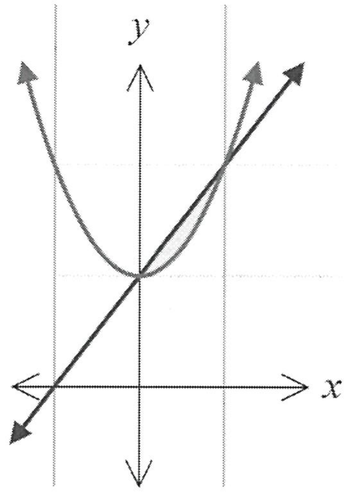
$= \ln \sqrt{\frac{|x-1|}{|x+1|}} + c$ ✓

Section 2 – Calculator assumed 20 marks

4. [3 marks]

Determine the volume generated when the region enclosed by the curves $y = x + 1$ and $y = x^2 + 1$, as shown, is rotated around the x-axis.

$$V_x = \pi \int_0^1 (x+1)^2 - (x^2+1)^2 \, dx \quad ✓ ✓$$



$$= \frac{7\pi}{15} \quad \text{or} \quad 1.466 \text{ units}^3$$

✓

5. [5 marks – 2, 2 and 1]

Fluid flow through a narrow pipe has been modelled by the equation $F = kr^{\frac{3}{2}}$ where F is the flow, k a constant and r the radius.

(a) How will the rate of flow change if the radius is increased by 44%?

$r = 1.44 r_0$ ✓

$\Rightarrow F = 1.44^{\frac{3}{2}} F_0$

$= 1.728 F_0$ ✓ i.e. 72.8% increase

(b) Use the incremental technique to estimate the change of radius that produces a 10% decrease in flow.

$\frac{\Delta F}{F} = \frac{3}{2} \frac{\Delta r}{r}$ ✓

$\Rightarrow \frac{\Delta r}{r} = \frac{2}{3}(-10) = -6\frac{2}{3}$ i.e. a $6\frac{2}{3}$% decrease ✓

(c) Is the incremental technique appropriate in (a)? Explain.

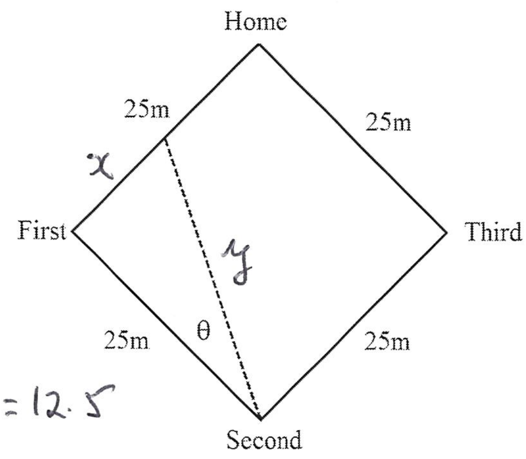
No, 44% is not incremental. It is too large.

✓

6. [7 marks – 4 and 3]

A baseball diamond consists of a 25 metre square.

A batter goes from the home plate and runs directly towards first base at 8 metres per second.



(a) How fast is his distance from second base changing when he is half-way to first base?

$\frac{dx}{dt} = -8$ Find $\frac{dy}{dx}$ at $x = 12.5$

✓ set up

$y^2 = x^2 + 25$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$ ✓ $\Rightarrow \frac{dy}{dx} = \frac{x}{y}$ ✓ $\cdot \frac{dx}{dt} = \frac{1}{\sqrt{5}} \cdot -8$

ie decreasing at $\frac{8}{\sqrt{5}} = 3.58$ m sec^{-1}. ✓

(b) How is angle θ changing at the same instant?

$\tan\theta = \frac{x}{25}$ ✓

$\therefore \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt} \cdot \frac{1}{25}$ ✓

$\frac{d\theta}{dt} = -8 \cdot \frac{1}{25} \cdot \frac{4}{5} = -\frac{32}{125}$ ✓

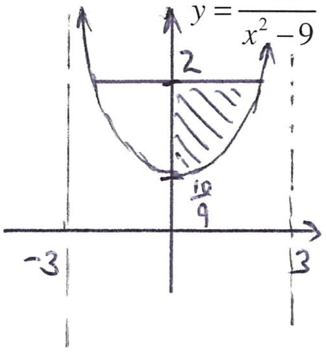
$\uparrow$

$\cos^2\theta = \left(\frac{25}{y}\right)^2 = \left(\frac{2}{\sqrt{5}}\right)^2$

ie at $-\frac{32}{125}$ radians per second. (0.256 rad/sec)

7. [5 marks]

Calculate the exact volume generated when the region contained between $y = 2$ and $y = \dfrac{-10}{x^2 - 9}$ for $-3 < x < 3$ is rotated about the y-axis.



✓

$V_x = \pi \int_a^b y^2\,dx$ becomes $V_y = \pi \int_c^d x^2\,dy$ ✓

$y = \dfrac{-10}{x^2 - 9}$

$x^2 - 9 = \dfrac{-10}{y}$

$\therefore V = \pi \displaystyle\int_{\frac{10}{9}}^{2} -\frac{10}{y} + 9\,dy$ ✓✓

$= \pi \left(-10 \ln y + 9y\right)\Big|_{\frac{10}{9}}^{2}$

$= \pi \left(-10 \ln 2 + 18 + 10 \ln \frac{10}{9} - 10\right)$

$= \pi \left(8 + 10 \ln \frac{5}{9}\right)$ ✓

(ClassPad gives [illegible])

or $\left(10 \ln 5 - 20 \ln 3 + 8\right)\pi$ ✓